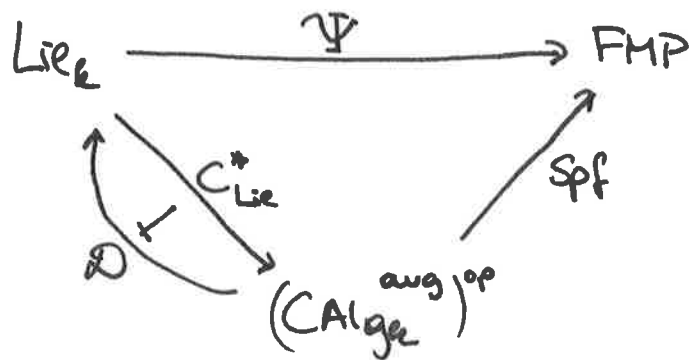


Where we are:



Observe that: $\Psi(g)(A) \simeq \text{CAlg}_k^{\text{aug}}(C_{\text{Lie}}^*(g), A) \simeq \text{Lie}_k(\mathcal{D}(A), g)$

$\Rightarrow \Psi$ preserves limits.

\Rightarrow Adjoint functor theorem tells us that there is a left adjoint Φ to Ψ (uses accessibility...)

Recall: an adjunction is an equivalence of ∞ -categories iff $\Phi \dashv \Psi$

① Ψ detects equivalences

② the unit $\eta: \text{id} \xRightarrow{\text{FMP}} \Psi \circ \Phi$ is an equivalence

Prop For $f: g \rightarrow h$ in Lie_k , if $\Psi(f)$ is an equivalence, then f is an equivalence.

Proof: $\Psi(g)(k \oplus k[n]) \simeq \Psi(h)(k \oplus k[n])$ by hypothesis

$$\text{Lie}_k(\mathcal{D}(k \oplus k[n]), g) \simeq \text{Lie}_k(\mathcal{D}(k \oplus k[n]), h)$$

Will use (+ prove) the following

Fact: $\mathcal{D}(k \oplus k[n]) \simeq \text{Free}_{\text{Lie}}(k[-1-n])$

~~more plausible~~ plausible b/c $C_{\text{Lie}}^*(\downarrow) \simeq k \oplus (k[-1-n])^\vee[-]$

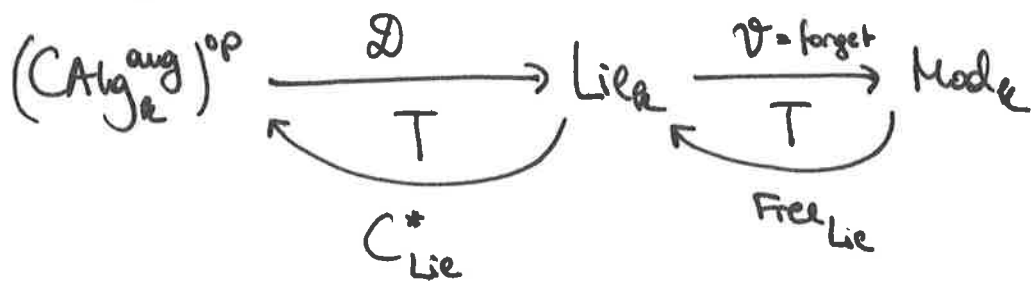
$$\begin{array}{c} \vdots \\ 12 \\ \text{Lie}_k(\text{Free}_{\text{Lie}}(k[-1-n]), g) \\ 12 \\ \text{Mod}_k(k[-1-n], g) \end{array}$$

$$\begin{array}{c} 12 \\ \text{Lie}_k(\text{Free}_{\text{Lie}}(k[-1-n]), h) \\ 12 \\ \text{Mod}_k(k[-1-n], h) \end{array}$$

$\Rightarrow \mathcal{V}(g) \xrightarrow{\mathcal{V}(f)} \mathcal{V}(h)$ is an equivalence ▣

So, just need to understand what \mathcal{D} does better.

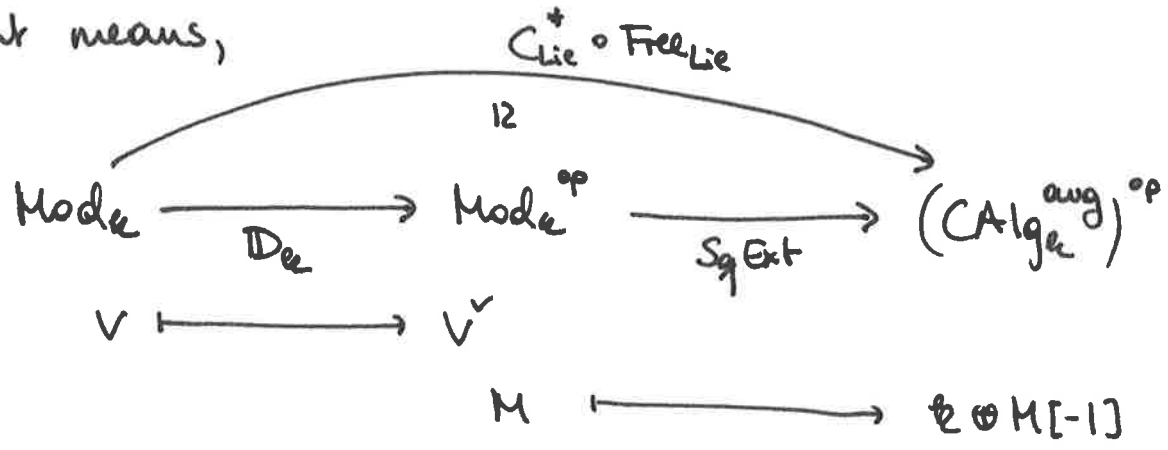
Strategy: Study the linear shadow, i.e. $\mathcal{V} \circ \mathcal{D}$:



We know there's a natural equivalence (Prop 2 from last time)

$$C_{\text{Lie}}^*(\text{Free}_{\text{Lie}}(V)) \cong k \oplus V^{\vee}[-1]$$

So that means,



Construct/recognize adjoints:

- \mathbb{D}_k is left adjoint to itself:

$$\text{Mod}_k(V, \mathbb{D}_k(W)) \cong \text{Mod}_k(V \oplus W, k) \cong \text{Mod}_k(W, \mathbb{D}_k(V)) \cong \text{Mod}_k^{\text{op}}(\mathbb{D}_k(V), W)$$

- There is a left adjoint L to $SqExt$ which we will analyze in some detail:

it's a case of the cotangent complex!

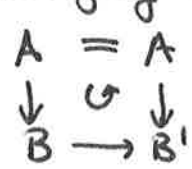
$\Rightarrow L \circ D_k \longrightarrow \mathcal{O} \circ D$, will use this for Fact.

The cotangent complex

Fix a char 0 field k . Let $CAlg_k^{dg}$ denote the model category of cdgas over k .

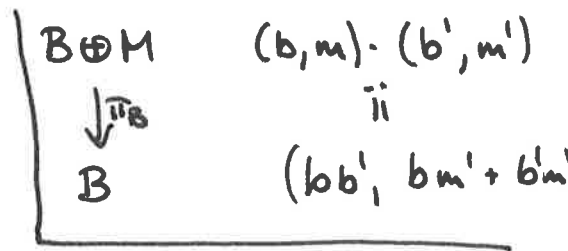
For $A \in CAlg_k^{dg}$, let $CAlg_A^{dg}$ denote the model category of A -algebras, $A = A$

For $B \in CAlg_A^{dg}$, let $(CAlg_A^{dg})/B$ denote the model category of A -algebras equipped w/ a map \twoheadrightarrow to B



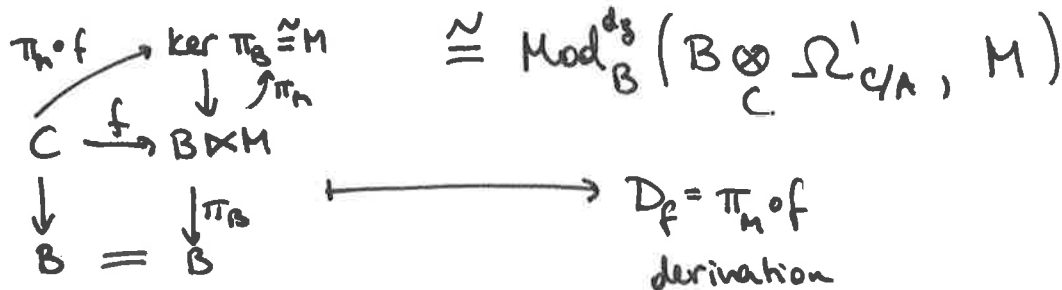
The functor $SqExt: \begin{cases} Mod_B^{dg} \longrightarrow (CAlg_A^{dg})/B \\ M \longmapsto B \times M \end{cases}$

is well-defined.



Now:

$$CAlg_{/B}^{dg}(C, B \times M) \cong Der_A(C, M)$$



Recall: $\Omega^1_{C/A} = J/J^2$ ← the "indecomposables", $J = \ker (C \otimes_A C \xrightarrow{m_C} C)$ [4]

Ex: $A \xrightarrow{E} k$, $\ker E = m_A$, $\Omega^1_{k/k} = m_A/m_A^2$, Zariski cotangent space

$$\text{Mod}_B^{dg}(B \otimes_C \Omega^1_{C/A}, M) \cong \text{Mod}_C^{dg}(\Omega^1_{C/A}, M) \xrightarrow{\cong} \text{Der}_A(C, M)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$g \longmapsto g \circ ddr$$

$$ddr: C \longrightarrow \Omega^1_{C/A} \text{ universal derivation}$$

$$x \longmapsto [x \otimes 1 - 1 \otimes x]$$

Prop: There is a Quillen adjunction

$$B_m \otimes_{-} \Omega^1_{-/A} : (CA_{lg}^{dg}_A)_{/B} \rightleftarrows \text{Mod}_B^{dg} : \text{Sq Ext}$$

Def'n The cotangent complex $\mathbb{L}_{B/A}$ is any B -module quasi-isomorphic to $B \otimes_P \Omega^1_{P/A}$, where $P \xrightarrow{\cong} B$ is a cofibrant replacement of B in $CA_{lg}^{dg}_A$.

trivial example: Let $A = k$, $B = k[x_1, \dots, x_n] = \text{Sym}(\{x_1, \dots, x_n\})$
Then B is free and therefore cofibrant, so

$$\mathbb{L}_{B/A} \cong \Omega^1_{B/A} = \bigoplus_{i=1}^n k[x_1, \dots, x_n] dx_i$$

Remark: For more complicated algebras B , you usually construct a "semi-free resolution"

$$P = \left(k[x_\alpha]_{\alpha \in I}, \partial = \sum_{\alpha} f_\alpha \frac{\partial}{\partial x_\alpha} \right) \xrightarrow{\cong} B$$

Lemma: (flat base change)

Consider a pushout square in $\text{Cat}_{\text{Alg}}^{\text{dg}}_A$:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow \phi & & \downarrow \\
 B & \longrightarrow & B' \cong A' \otimes_A B
 \end{array}$$

where one of f or ϕ is flat.

Then,

$$B' \otimes_A \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B'/A'}$$

is a quasi-isomorphism as a map of B' -modules.

Pf: We have base change for Ω' . Choose a cofibrant replacement

$P \rightarrow B$ of B as an A -algebra. Then

$$P' := P \otimes_A A' \longrightarrow B \otimes_A A' = B'$$

is a cofibrant replacement of B' and the map is a w.e. by flatness.

Then the base change map (for Ω')

$$\begin{array}{ccc}
 B' \otimes_B (B \otimes_P \Omega'_{P/A}) & \longrightarrow & B' \otimes_{P'} \Omega'_{P'/A'} \\
 \cong \downarrow & & \cong \downarrow \\
 B' \otimes_B \mathbb{L}_{B/A} & & \mathbb{L}_{B'/A'}
 \end{array}$$

is a weak equivalence. □

Prop (transitivity) For $A \rightarrow B \rightarrow C$ in CAlg_k^{dg} , the natural maps

$$C \otimes_B \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B}$$

form a cofiber sequence.

Application: Consider an augmented k -algebra $k \rightarrow A \rightarrow k$.

Then

$$\text{CAlg}_k^{aug} = (\text{CAlg}_k)_{/k} \begin{matrix} \xleftarrow{\text{SgExt (w/ shift)}} \\ \text{T} \\ \xrightarrow{\text{Mod}_k} \\ \text{L} \end{matrix}$$

$$A \longmapsto \left(k \otimes_A \mathbb{L}_{A/k} \right) [1]$$

By transitivity proposition, $\mathbb{L}_{k/A}$.

Hence, $v \cdot \mathcal{D} \simeq \mathcal{D}_k \cdot L = A \longmapsto \mathbb{L}_{k/A}^v$.

Lemma For $\mathfrak{g} \in \text{Lie}_k$ s.t.

(a) $\mathfrak{g}^d = 0$ for $d \leq 0$
 $\begin{matrix} -n & \dots & 0 & 1 & 2 & \dots \\ 0 & \dots & 0 & \mathfrak{g}^1 & \mathfrak{g}^2 & \dots \end{matrix}$

(b) $\dim_k \mathfrak{g}^d < \infty$ for all d

the unit map $\mathfrak{g} \longrightarrow \mathcal{D}(\text{C}_{\text{Lie}}^*(\mathfrak{g}))$ is an equivalence.

Corollary For $n \geq 0$, $\mathfrak{g}^{(n)} := \text{Free}_{\text{Lie}}(k[-1-n])$ satisfies (a) & (b),

so $\mathfrak{g}^{(n)} \xrightarrow{\simeq} \mathcal{D}(\text{C}_{\text{Lie}}^*(\mathfrak{g}^{(n)})) = k \oplus k[n]$

\Rightarrow FACT needed above !!

Proof of Lemma: It suffices to prove that the unit map is an equivalence of the underlying cochain complexes.

Now, the map

$$\mathcal{V}(\mathfrak{g}) \longrightarrow \mathcal{V}(\mathcal{D} \circ C_{\text{Lie}}^*(\mathfrak{g})) \cong \mathbb{L}_{\mathbb{k}}^{\vee} / C_{\text{Lie}}^*(\mathfrak{g})$$

has a predual

$$\mathbb{L}_{C_{\text{Lie}}^*(\mathfrak{g})/\mathbb{k}} \otimes_{C_{\text{Lie}}^*(\mathfrak{g})} \mathbb{k} \longrightarrow \mathfrak{g}^{\vee}[-1]$$

Recall that as a graded algebra,

$$C_{\text{Lie}}^*(\mathfrak{g})^* = \widehat{\text{Sym}}(\mathfrak{g}^{\vee}[-1]) = \prod_{n \geq 0} \text{Sym}^n(\mathfrak{g}^{\vee}[-1])$$

not quite free b/c it's the completed symmetric algebra

Fix a basis $\{x_1, \dots, x_n\}$ for $(\mathfrak{g}^1)^{\vee}$, so that

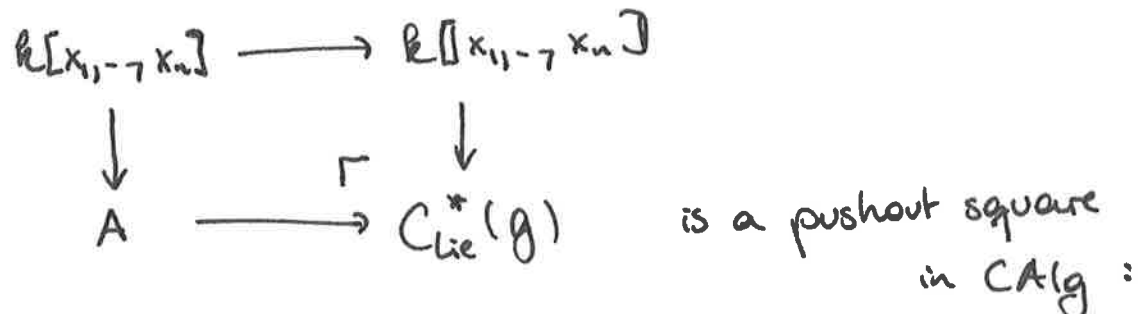
$$C_{\text{Lie}}^0(\mathfrak{g}) \cong \mathbb{k}\langle\langle x_1, \dots, x_n \rangle\rangle$$

Let $A = \left(\bigoplus_{n \geq 0} \text{Sym}^n(\mathfrak{g}^{\vee}[-1]), d \right) \hookrightarrow C_{\text{Lie}}^*(\mathfrak{g})$
differential induced by inclusion

$$\text{So, } A^0 = \mathbb{k}\langle\langle x_1, \dots, x_n \rangle\rangle \xrightarrow{\phi^0} \mathbb{k}\langle\langle x_1, \dots, x_n \rangle\rangle = C_{\text{Lie}}^0(\mathfrak{g}).$$

By hypothesis construction

$$A \otimes_{\mathbb{k}\langle\langle x_1, \dots, x_n \rangle\rangle} \mathbb{k}\langle\langle x_1, \dots, x_n \rangle\rangle \xrightarrow{\cong} C_{\text{Lie}}^*(\mathfrak{g}), \text{ and}$$



As ϕ^0 is flat, so

$$H^n(A) \otimes_{k[\dots]} k \cong H^n(C_{\text{Lie}}^*(g)).$$

As ϕ^0 is flat, by flat base change,

$$\mathbb{L}_{C_{\text{Lie}}^*(g)/A} \otimes_{C_{\text{Lie}}^*(g)} k \simeq \mathbb{L}_{k[x_1, \dots, x_n]/k[x_1, \dots, x_n]} \otimes k \simeq \mathbb{L}_k/k$$

where $R = k[x_1, \dots, x_n] \otimes_{k[x_1, \dots, x_n]} k \simeq k \Rightarrow \mathbb{L}_{R/k} \simeq 0.$

Hence by the cofiber sequence of transitivity, we can replace

$$\begin{array}{ccc} \mathbb{L}_{C_{\text{Lie}}^*(g)/k} \otimes_{C_{\text{Lie}}^*(g)} k & \simeq & \mathbb{L}_{A/R} \otimes_A k \\ \downarrow \text{predual map} & & \downarrow \text{12} \\ g^v[-1] & & g^v[-1] \end{array}$$

A is semi-free, so cofibrant, so can compute as Ω^1 .

