Where we are:

$$
\begin{tikzcd}
\mathrm{Lie}_k \arrow[rr, "\Psi"] \arrow[dr, "C^*_{\mathrm{Lie}}"] & & \mathrm{FMP} \\
& (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \arrow[ul, bend left, "\mathfrak{D}"] \arrow[ur, "\mathrm{Spf}"'] &
\end{tikzcd}
$$

Observe that: $\Psi(\mathfrak{g})(A) \simeq \mathrm{CAlg}_k^{\mathrm{aug}}(C^*_{\mathrm{Lie}}(\mathfrak{g}), A) \simeq \mathrm{Lie}_k(\mathfrak{D}(A), \mathfrak{g})$

$\Longrightarrow$ Ψ preserves limits.

$\Longrightarrow$ Adjoint functor theorem tells us that there is a left adjoint Φ to Ψ (uses accessibility...)

<u>Recall</u>: an adjunction is an equivalence of ∞-categories iff

$$\Phi \dashv \Psi$$

① Ψ detects equivalences

② the unit $\eta: \mathrm{id}_{\mathrm{FMP}} \Longrightarrow \Psi \circ \Phi$ is an equivalence

<u>Prop</u> For $f: \mathfrak{g} \to \mathfrak{h}$ in Lie_k, if $\Psi(f)$ is an equivalence, then f is an equivalence.

<u>Proof</u>: $\Psi(\mathfrak{g})(k \oplus k[n]) \simeq \Psi(\mathfrak{h})(k \oplus k[n])$ by hypothesis

$$
\begin{array}{ccc}
\Psi(\mathfrak{g})(k \oplus k[n]) & \simeq & \Psi(\mathfrak{h})(k \oplus k[n]) \\
\wr\wr & & \wr\wr \\
\mathrm{Lie}_k(\mathfrak{D}(k \oplus k[n]), \mathfrak{g}) & \simeq & \mathrm{Lie}_k(\mathfrak{D}(k \oplus k[n]), \mathfrak{h}) \\
\vdots & & \vdots
\end{array}
$$

Will use (+ prove) the following

<u>FACT</u>: $\mathfrak{D}(k \oplus k[n]) \simeq \mathrm{Free}_{\mathrm{Lie}}(k[-1-n])$

~~[illegible]~~ plausible b/c $C^*_{\mathrm{Lie}}(\quad) \simeq k \oplus (k[-1-n]^\vee)[-1]$

$$
\begin{array}{cc}
\vdots & \\
\wr\wr & \wr\wr \\
\mathrm{Lie}_k(\mathrm{Free}_{\mathrm{Lie}}(k[-1-n]), \mathfrak{g}) & \mathrm{Lie}_k(\mathrm{Free}_{\mathrm{Lie}}(k[-1-n]), \mathfrak{h}) \\
\wr\wr & \wr\wr \\
\mathrm{Mod}_k(k[-1-n], \mathfrak{g}) & \mathrm{Mod}_k(k[-1-n], \mathfrak{h})
\end{array}
$$

$\Longrightarrow$ $\vartheta(\mathfrak{g}) \xrightarrow{\vartheta(f)} \vartheta(\mathfrak{h})$ is an equivalence ∎

So, just need to understand what $\mathfrak{D}$ does better.

<u>Strategy</u>: Study the linear shadow, i.e. $\vartheta \circ \mathfrak{D}$:

$$
\begin{tikzcd}
(\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}} \arrow[r, shift left, "\mathfrak{D}"] & \mathrm{Lie}_k \arrow[l, shift left, bend left, "C^*_{\mathrm{Lie}}"] \arrow[r, shift left, "\vartheta = \text{forget}"] & \mathrm{Mod}_k \arrow[l, shift left, bend left, "\mathrm{Free}_{\mathrm{Lie}}"]
\end{tikzcd}
$$

We know there's a natural equivalence (Prop 2 from last time)

$$C^*_{\mathrm{Lie}}(\mathrm{Free}_{\mathrm{Lie}}(V)) \simeq k \oplus V^\vee[-1]$$

So that means,

$$
\begin{tikzcd}
\mathrm{Mod}_k \arrow[r, "\mathbb{D}_k"'] \arrow[rr, bend left, "C^*_{\mathrm{Lie}} \circ \mathrm{Free}_{\mathrm{Lie}}", "\wr\wr"'] & \mathrm{Mod}_k^{\mathrm{op}} \arrow[r, "\mathrm{SqExt}"'] & (\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}} \\
V \arrow[r, mapsto] & V^\vee & \\
& M \arrow[r, mapsto] & k \oplus M[-1]
\end{tikzcd}
$$

Construct/recognize adjoints:

- $\mathbb{D}_k$ is left adjoint to itself:

$$
\begin{array}{c}
\mathrm{Mod}_k(V, \mathbb{D}_k(W)) \simeq \mathrm{Mod}_k(V \otimes W, k) \simeq \mathrm{Mod}_k(W, \mathbb{D}_k(V)) \\
\hfill \| \\
\hfill \mathrm{Mod}_k^{\mathrm{op}}(\mathbb{D}_k(V), W)
\end{array}
$$

- There is a left adjoint L to SqExt which we will analyze in some detail:

  it's a case of the cotangent complex!

$\Longrightarrow$ $L \circ \mathbb{D}_k \dashv \Theta \circ \mathcal{D}$, will use this for Fact.

<u>The cotangent complex</u>

Fix a char 0 field k. Let CAlg_k^{dg} denote the model category of cdga's over k.

For $A \in \mathrm{CAlg}_k^{dg}$, let CAlg_A^{dg} denote the model category of A-algebras,
$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

For $B \in \mathrm{CAlg}_A^{dg}$, let $\left(\mathrm{CAlg}_A^{dg}\right)_{/B}$ denote the model category of A-algebras equipped w/ a map ~~—~~ to B

The functor SqExt: $\begin{cases} \mathrm{Mod}_B^{dg} \longrightarrow \left(\mathrm{CAlg}_A^{dg}\right)_{/B} \\ M \longmapsto \underbrace{B \ltimes M} \end{cases}$

is well-defined.

$$\boxed{\begin{array}{ll} B \oplus M & (b,m)\cdot(b',m') \\ \downarrow \pi_B & \quad\; \shortparallel \\ B & (bb', bm' + b'm) \end{array}}$$

<u>Now:</u> $\mathrm{CAlg}^{dg}_{/B}\left(C, B \ltimes M\right) \cong \mathrm{Der}_A(C, M)$

$$\cong \mathrm{Mod}_B^{dg}\left(B \underset{C}{\otimes} \Omega^1_{C/A}, M\right)$$

$$\begin{tikzcd} & \ker \pi_B \cong M \arrow[d] \\ C \arrow[ur, bend left, "\pi_M \circ f"] \arrow[r, "f"] \arrow[d] & B \ltimes M \arrow[u, bend right, "\pi_M"'] \arrow[d, "\pi_B"] \\ B \arrow[r, equal] & B \end{tikzcd} \longmapsto \underset{\text{derivation}}{D_f = \pi_M \circ f}$$

Recall: $\Omega^1_{C/A} = J/J^2$, $J = \ker(C \otimes_A C \xrightarrow{m_C} C)$

the "indecomposables"

Ex: $A \xrightarrow{\varepsilon} k$ $\quad \Omega^1_{A/k} = m_A/m_A^2$, Zariski cotangent space

$\ker \varepsilon = m_A$

$$\mathrm{Mod}_B^{dg}(B \otimes_C \Omega^1_{C/A}, M) \cong \mathrm{Mod}_C^{dg}(\Omega^1_{C/A}, M) \xrightarrow{\cong} \mathrm{Der}_A(C, M)$$

$$g \longmapsto g \circ d_{dR}$$

$$d_{dR} : C \longrightarrow \Omega^1_{C/A} \quad \text{universal derivation}$$

$$x \longmapsto [x \otimes 1 - 1 \otimes x]$$

Prop: There is a Quillen adjunction

$$B \otimes_{-} \Omega^1_{-/A} : (\mathrm{CAlg}_A^{dg})_{/B} \rightleftarrows \mathrm{Mod}_B^{dg} : \mathrm{SqExt}$$

Def'n The cotangent complex $\mathbb{L}_{B/A}$ is any B-module quasi-isomorphic to $B \otimes_P \Omega^1_{P/A}$, where $P \xrightarrow{\simeq} B$ is a cofibrant replacement of B in CAlg_A^{dg}.

trivial example: Let $A = k$, $B = k[x_1, \dots, x_n] = \mathrm{Sym}(\{x_1, \dots, x_n\})$

Then B is free and therefore cofibrant, so

$$\mathbb{L}_{B/A} \simeq \Omega^1_{B/A} = \bigoplus_{i=1}^{n} k[x_1, \dots, x_n]\, dx_i$$

Remark: For more complicated algebras B, you usually construct a "semi-free resolution"

$$P = \left(k[x_\alpha]_{\alpha \in I},\ \partial = \sum_\alpha f_\alpha \frac{\partial}{\partial x_\alpha}\right) \xrightarrow{\simeq} B$$

Lemma: (flat base change)

Consider a pushout square in CAlg^{dg}_A:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & A' \arrow[d] \\ B \arrow[r] & B' \cong A' \otimes_A B \end{tikzcd}$$

where one of f or ϕ is flat.

Then,

$$B' \otimes_A \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B'/A'}$$

is a quasi-isomorphism as a map of B'-modules.

Pf: We have base change for Ω^1. Choose a cofibrant replacement $P \to B$ of B as an A-algebra. Then

$$P' := P \otimes_A A' \longrightarrow B \otimes_A A' = B'$$

is a cofibrant replacement of B' and the map is a w.e. by flatness.

Then the base change map (for Ω^1)

$$\begin{array}{ccc} B' \otimes_B (B \otimes_P \Omega^1_{P/A}) & \longrightarrow & B' \otimes_{P'} \Omega^1_{P'/A'} \\ \wr\wr & & \wr\wr \\ B' \otimes_B \mathbb{L}_{B/A} & & \mathbb{L}_{B'/A} \end{array}$$

is a weak equivalence. ∎

Prop (transitivity) For $A \to B \to C$ in CAlg_k^{dg}, the natural maps

$$C \otimes_B \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B}$$

form a cofiber sequence.

Application: Consider an augmented k-algebra $k \to A \to k$.

Then

$$\begin{tikzcd} \mathrm{CAlg}_k^{aug} = (\mathrm{CAlg}_k)_{/k} \arrow[rr, bend right, "L"'] \arrow[rr, phantom, "T"] & & \mathrm{Mod}_k \arrow[ll, bend right, "\underbrace{\mathrm{SqExt}}\ (\text{w/ shift})"'] \end{tikzcd}$$

$$A \longmapsto \left(k \otimes_A \mathbb{L}_{A/k}\right)[1]$$

By transitivity proposition, $\qquad \cong \mathbb{L}_{k/A}$.

Hence, $\vee \circ \mathfrak{D} \simeq D_k \circ L : \quad A \longmapsto \mathbb{L}_{k/A}^{\vee}$.

Lemma For $\mathfrak{g} \in \mathrm{Lie}_k$ s.t.

ⓐ $\mathfrak{g}^d = 0$ for $d \leq 0$

$$\begin{array}{cccccccc} -n & \cdots & 0 & 1 & 2 & \cdots \\ 0 & \cdots & 0 & \mathfrak{g}^1 & \mathfrak{g}^2 & \cdots \end{array}$$

ⓑ $\dim_k \mathfrak{g}^d < \infty$ for all d

the unit map $\quad \mathfrak{g} \longrightarrow \mathfrak{D}(C^*_{\mathrm{Lie}}(\mathfrak{g}))$ is an equivalence.

Corollary For $n \geq 0$, $\mathfrak{g}^{(n)} := \mathrm{Free}_{\mathrm{Lie}}(k[-1-n])$ satisfies ⓐ & ⓑ, so

$$\mathfrak{g}^{(n)} \xrightarrow{\simeq} \mathfrak{D}\Big(\underbrace{C^*_{\mathrm{Lie}}(\mathfrak{g}^{(n)})}_{= k \oplus k[n]}\Big)$$

$\Longrightarrow$ FACT needed above :)

Proof of Lemma: It suffices to prove that the unit map is an equivalence of the underlying cochain complexes.
Now, the map

$$\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathcal{D} \circ C^*_{Lie}(\mathfrak{g})) \simeq \mathbb{L}^\vee_{k/C^*_{Lie}(\mathfrak{g})}$$

has a predual

$$\mathbb{L}_{C^*_{Lie}(\mathfrak{g})/k} \underset{C^*_{Lie}(\mathfrak{g})}{\otimes} k \longrightarrow \mathfrak{g}^\vee[-1]$$

Recall that as a graded algebra,

$$C^*_{Lie}(\mathfrak{g})^* = \widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1]) = \prod_{n\geq 0} \mathrm{Sym}^n(\mathfrak{g}^\vee[-1])$$

not quite free b/c it's the completed symmetric algebra

Fix a basis $\{x_1, \dots, x_n\}$ for $(\mathfrak{g}^1)^\vee$, so that

$$C^0_{Lie}(\mathfrak{g}) \cong k[[x_1, \dots, x_n]]$$

Let $$A = \Big(\bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathfrak{g}^\vee[-1]), d\Big) \hookrightarrow C^*_{Lie}(\mathfrak{g})$$

differential induced by inclusion

So, $$A^0 = k[x_1, \dots, x_n] \overset{\phi^0}{\hookrightarrow} k[[x_1, \dots, x_n]] = C^0_{Lie}(\mathfrak{g}).$$

By ~~hypothesis~~, construction, $$A \underset{k[x_1, \dots, x_n]}{\otimes} k[[x_1, \dots, x_n]] \xrightarrow{\cong} C^*_{Lie}(\mathfrak{g}),$$ and

$$\begin{tikzcd}
k[x_1, \dots, x_n] \arrow[r] \arrow[d] & k[[x_1, \dots, x_n]] \arrow[d] \\
A \arrow[r] & C^*_{Lie}(\mathfrak{g}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout square in CAlg :

As ϕ° is flat, so

$$H^n(A) \underset{k[\dots]}{\otimes} k[\![\dots]\!] \cong H^n(C^*_{Lie}(g)).$$

As ϕ° is flat, by flat base change,

$$\mathbb{L}_{C^*_{Lie}(g)/A} \underset{C^*_{Lie}(g)}{\otimes} k \simeq \mathbb{L}_{k[\![x_1,\dots,x_n]\!]/k[x_1,\dots,x_n]} \otimes k \simeq \mathbb{L}_{R/k},$$

where $R = k[\![x_1,\dots,x_n]\!] \underset{k[x_1,\dots,x_n]}{\otimes} k \simeq k \Rightarrow \mathbb{L}_{R/k} \simeq 0.$

Hence by the cofiber sequence of transitivity, we can replace

$$\begin{tikzcd} \mathbb{L}_{C^*_{Lie}(g)/k} \underset{C^*_{Lie}(g)}{\otimes} k \arrow[r, phantom, "\simeq"] \arrow[d, "\text{predual map}"] & \mathbb{L}_{A/k} \underset{A}{\otimes} k \arrow[d, phantom, "\cong" rotate=90] \\ g^\vee[-1] & g^\vee[-1] \end{tikzcd}$$

A is semi-free, so cofibrant, so can compute as Ω^1.